# Building Representative Matched Samples with Multi-valued Treatments in Large Observational Studies*

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**Abstract**

In this paper, we present a new way of matching in observational studies that overcomes three limitations of existing matching approaches. First, it directly balances covariates with multi-valued treatments without explicitly estimating the generalized propensity score. Second, it builds self-weighted matched samples that are representative of a target population by design. Third, it can handle large data sets, with hundreds of thousands of observations, in a couple of minutes. The key insights of this new approach to matching are balancing the treatment groups relative to a target population and positing a linear-sized mixed integer formulation of the matching problem. We formally show that this formulation is more effective than alternative quadratic-sized formulations, as its reduction in size does not affect its strength from the standpoint of its linear programming relaxation. We also show that this formulation can be used for matching with distributional covariate balance in polynomial time under certain assumptions on the covariates and that it can handle large data sets in practice even when the assumptions are not satisfied. This algorithmic characterization is key to handling large data sets. We illustrate this new approach to matching in both a simulation study and an observational study of the impact of an earthquake on educational attainment. With this approach, the results after matching can be visualized with simple and transparent graphical displays: while increasing levels of exposure to the earthquake have a negative impact on school attendance, there is no effect on college admission test scores.

*Keywords:* Causal Inference; Multi-valued Treatment; Observational Studies; Optimal Matching; Propensity Score; Representative Study.

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# 1 Introduction

## 1.1 Practical appeal of matching

In observational studies, matching is a general method for covariate adjustment that approximates the ideal experiment that would be conducted if controlled experimentation was possible. Under the assumption of strong ignorability (Rosenbaum and Rubin 1983), or no unmeasured confounders (Imbens 2004), matching methods are often used to estimate treatment effects in observational studies with binary treatments (e.g., Dehejia and Wahba 1999; Haviland et al. 2007), longitudinal (Lu 2005; Zubizarreta et al. 2014) and multilevel data (Li et al. 2013; Zubizarreta and Keele 2017), but also under different assumptions, for example with instrumental variables (Baiocchi et al. 2010; Zubizarreta et al. 2013b) or in discontinuity designs (Keele et al. 2015; Mattei and Mealli 2016).

The practical appeal of matching methods lies in part in the conceptual simplicity and transparency of their adjustments (Cochran and Rubin 1973). These adjustments are an interpolation instead of an extrapolation based on a model that can be misspecified (Rosenbaum 1987a). Matching also enables the integration of quantitative and qualitative analyses (Rosenbaum and Silber 2001). Furthermore, since matching methods do not routinely use the outcomes for their adjustments, it is often argued that they promote the objectivity of the study by separating the design and analysis of an observational study into two distinct stages (Rubin 2008). Finally, matching methods can facilitate simpler forms of statistical inference and sensitivity analyses to hidden biases (see, for instance, Chapter 3 of Rosenbaum 2010). See Stuart (2010), Imbens (2015), and Rosenbaum (2019) for overviews of matching methods.

## 1.2 Three challenges

Most work in matching has been for binary treatments and, although there are methods for more general treatments (e.g., Lu et al. 2001, 2011; Yang et al. 2016; Lopez et al. 2017), some challenges remain. One challenge relates to the difficulty of balancing covariates by means of the estimated propensity score (Rosenbaum and Rubin 1983) or generalizations thereof (e.g.,

Joffe and Rosenbaum 1999; Imbens 2000; Imai and Van Dyk 2004). The propensity score is the conditional probability of treatment assignment given observed covariates. It has the important property that matching on the propensity score tends to balance the covariates used to estimate the score; however, in any given data set it might be difficult to balance the covariates even if the propensity score model is correctly specified (Yang et al. 2012). This challenge has been discussed, e.g., by Hill (2011) and Zubizarreta et al. (2011), and alternative methods that directly balance the covariates have been proposed (e.g., Diamond and Sekhon 2013; Zubizarreta 2012), yet these methods are for binary treatments. The difficulties of balancing covariates by matching on generalizations of the propensity score can easily be exacerbated with multi-valued treatments.

The second challenge with matching methods, both for binary and multi-valued treatments, is targeting parameters of general scientific and policy interest, especially when there is limited overlap in covariate distributions across treatment groups. When there is sufficient overlap, it is possible to estimate the average effect of one treatment in place of another treatment (possibly the control treatment) on one of these two treatment groups, but it is difficult to estimate other parameters such as the average effect on all treatment groups (in resemblance to the overall average treatment effect), or on a particular group as defined by covariates (the conditional average treatment effect), without matching with replacement or reweighting, either of which can make inference more complicated, as discussed by Abadie and Imbens (2008). When there is limited overlap, it is not possible to target the average effect of one treatment in place of another treatment for any of these treatment groups without imposing strong parametric assumptions, and it is common in practice to settle for more local average effects which may have more internal validity but be less generalizable. Often, these analyses are criticized for their limited scientific and policy interest (see, e.g., Imbens 2010).

A third challenge relates to matching in large data sets. Most common optimization-based matching methods rely on quadratic-sized formulations that cannot handle data sets with hundreds of thousands of observations quickly. As we describe below, the problem is that such formulations are too big to be practical or that they require additional structure on the

covariates in order to run quickly.

Through a case study of the impact of an earthquake on educational achievement, our goal in this paper is to overcome these three challenges and present a new matching method that: (i) handles multi-valued treatments and directly balances covariates without explicitly estimating a generalization of the propensity score; (ii) finds self-weighting matched samples that are not only balanced but also have a similar structure to that of a target population (thereby allowing the investigator to target parameters of general policy and scientific interest); and (iii) runs quickly in large data sets, for example, with hundreds of thousands of observations in a couple of minutes. For this, we use the following ideas.

## 1.3 Main ideas

Optimization-based methods for matching with binary treatments can be divided, roughly, into network flow methods (e.g., Rosenbaum 1989; Hansen 2004; Pimentel et al. 2015) and integer or mixed integer programming (MIP) methods (e.g., Nikolaev et al. 2013; Sauppe et al. 2014; Zubizarreta 2012; Zubizarreta et al. 2014). For the most, network flow methods minimize a measure of covariate distances between matched units, and not covariate balance directly, although there is the clever extension by Pimentel et al. (2015) for nested nominal covariates.[1] The advantage of MIP methods is that they target covariate balance (and distances) more directly and flexibly. However, MIP based methods require the solution of a theoretically intractable or NP-hard problem, whereas network flow matching methods only require the solution of a polynomially solvable problem that is tractable both in theory and practice. Still, thanks to state-of-the-art MIP solvers which nearly double their speeds every year (Bixby 2012; Achterberg and Wunderling 2013), MIP methods are mostly tractable in practice, and hence the computational advantage of network flow methods has been steadily decreasing.

[1]Throughout this paper, covariate balance refers to the extent to which covariate distributions are similar across treatment groups in *aggregate*, whereas covariate distances, to the similitude of *individual* observations across matched treatment groups. In practice, covariate balance is often measured in terms of absolute standardized differences in means (Rosenbaum and Rubin 1985), although other distributional forms of covariate balance may be desirable (see, e.g., Resa and Zubizarreta 2016). An example of a robust covariate distance is the rank-based Mahalanobis distance within propensity score calipers Rosenbaum (2010).

If we consider matching with three or more treatments or exposures, this computational advantage vanishes as network flow methods for minimum distance problems also become theoretically intractable (Michael and David 1979). In addition, the large size of standard MIP formulations for such problems also makes them extremely challenging in practice (Burkard and Cela 1999). To circumvent this computational limitation, we match each treatment group to a representative sample of a target population and develop a new MIP-based matching formulation. A key to develop this method is a linear programming (LP) view of the difference in computational complexity between minimum distance matching and MIP-based methods. This view models the central problem of both methods as a MIP problem and then analyses the strength of their LP relaxation (see Section 4 for details). For minimum distance matching the LP relaxation has the strongest possible *integral* property, which implies the associated MIP problem can be solved as an LP problem in polynomial time (Schrijver 2003). For MIP based techniques the LP relaxation fails to be integral, but having an LP relaxation that is "close" to integral is known to result on small solution times (Vielma 2015). Unfortunately, we often face a trade-off between strong, but large formulations and smaller, but weaker formulations. With this in mind, we show that a linear-sized MIP formulation for covariate balance is as strong as an alternative, and more common, quadratic-sized formulation. We also show that this formulation is in fact integral when only two covariates are considered or when the nested covariate structure of Pimentel et al. (2015) is present (see Section 4.3 for details). While the formulation is not integral when more than three covariates are used, the associated MIP nonetheless remains strong and can be solved in minutes in large data sets. In contrast, attempting to solve the problem with existing formulations (both network and quadratic-sized MIP formulations) results in an out-of-memory error even in a workstation with 32GB of RAM. In this age of big data, this algorithmic characterization is key to handling large data sets in a practical manner.

In addition to allowing a computationally practical approach for matching with multi-valued treatments, the use of this auxiliary reference sample also increases the versatility of the method. By selecting this reference or template sample in different ways, we can target different estimands beyond the common average treatment effect on the treated, such as the (general) average treatment effect or a conditional average treatment effect for a particular

group of the population (e.g., subjects with a particular age in a particular ethnic group). In this manner we can build representative matched samples with multi-valued treatments that may be numerous and unordered. In our procedure, covariate distances between matching play a secondary role (relegated to the rematching of the balanced units, in the spirit of Zubizarreta et al. 2014). In fact, our procedure does not require explicitly modeling the propensity score but rather directly balances the covariates by design; that is, as specified before matching by the investigator. Furthermore, it facilitates checking covariate balance, as one can tabulate and plot the distributions of the observed covariates as opposed to evaluating balance in the context of a model.

## 1.4 Case study and outline

We illustrate all these ideas in the context of an observational study of the impact of a natural disaster on educational achievement. In particular, we estimate the effect of increasing levels of exposure to the 2010 Chilean earthquake on the test scores of university admissions exams of senior high school students. This is a very important question because due to its location in the Pacific Ring of Fire, Chile is considered one of the most seismically active countries in the world (CFE-DM 2017), experiencing 78 earthquakes above 7.0 magnitude since 1900 (CSN Universidad de Chile 2018) and holding the record for the strongest earthquake ever registered (USGS 2018). Furthermore, despite its sustained economic growth and being one of the most robust economies in Latin America, Chile has the most unequal income distribution among the 35 OECD countries as measured by the Gini coefficient, above even Mexico and the United States (OECD 2016). In this context of extreme income inequality it is important to understand the impact of these recurring natural disasters on standardized university admissions examination scores as they almost fully determine students' entrance to university, one of the main avenues of social mobility and opportunity in Chile (Torche 2005).

To address this question, the rest of this paper is organized as follows. In Section 2, we describe the Chilean educational system, the 2010 Chilean earthquake, and a particularly rich data set that is a census of the same students before and after the earthquake. In

Section 3 we explain how to use the ideas in Silber et al. (2014) for building representative matched samples of target populations with possibly unordered and many treatment or exposure groups. In Section 4, we formally analyze and contrast the properties of linear- and quadratic-sized MIP formulations for distributional balance, and evaluate their performance in data sets of increasing size. In Section 5 we present estimates of the impact of levels of exposures to the earthquake both on school attendance and standardized test scores for university admission. In Section 6 we conclude with a summary and remarks.

# 2 Impact of a natural disaster on educational opportunity

## 2.1 On the Chilean educational system

In Chile, most students complete the required 12 years of school education (MDS 2013). In their last year of high school, students take a college admission test called *Prueba de Selección Universitaria* or PSU in order to apply to college. The PSU is a high-stakes test as it comprises nearly 80% of the final college admission score and can only be taken once a year, at the end of the Chilean academic year. Most students register for the test in their last year of high school, and the vast majority of them takes it only once in their lives. The importance of the PSU does not only relate to being admitted into a given college, but also to obtaining financial aid and having the possibility to actually attend college(Dinkelman and Martinez 2014). In Chile, higher education is one of the main avenues for social mobility and the improvement of later life outcomes (Torche 2005). Studies have shown that even though the returns to higher education in Chile are heterogeneous, there are still high, positive returns to attending college, especially highly-selective ones (Hastings et al. 2013; González-Velosa et al. 2015).

## 2.2 The 2010 Chilean earthquake

On February 27th, 2010, an earthquake of magnitude 8.8 struck near Concepción, Chile's second largest city, going down in history as the 6$^{th}$ most severe earthquake registered since 1900 thus far (USGS 2014). The disaster severely damaged over 500,000 homes and affected more than 2 million people. Losses amounted to 18% of the gross domestic product (MINEDUC 2013). Nearly 20% of the schools in the affected regions suffered moderate damages or worse, and many schools had to be completely reconstructed. In addition, over 40% of the students in these regions could not start their academic year on time. 24 million US dollars were redirected to rebuild and restore the affected schools and the academic calendar was adjusted (MINEDUC 2013).

## 2.3 A longitudinal census of students

In our analyses, we use a rich administrative data set of the same high school students measured before and after the earthquake. This data set is a longitudinal census as it comprises all Chilean students in 10th grade in 2008 (before the earthquake) and it collects their information again in 12th grade in 2010 (after the earthquake), when they are finishing high school and applying for college. In 2008, the data provides detailed (pre-exposure) measures of the students, their schools, and respective households, for which we will adjust for by matching. For instance, the data provides the standardized test scores from the Education Quality Measurement System (SIMCE), school attendance, GPA ranking within the school, and school characteristics such as its socioeconomic status. In addition, the data provides extensive socioeconomic information of the students and their households, such as their parents' education and the household income.

In 2010, the data contains two outcomes of interest: (i) the students' school attendance that academic year, and (ii) the language and mathematics PSU scores (both measured after the earthquake). The first outcome is assessed as a percentage. The scale of the PSU is the same for the two tests, ranging between 150 and 850 points, with a mean of 500 points and a standard deviation of 110 points. The data comprises 121279 students measured before

and after the earthquake.

### 2.4 Earthquake intensity levels

We use peak ground acceleration (PGA) to measure the strength of the earthquake. In contrast to other seismic intensity scales, for example, the Mercalli or the Richter scales, PGA is a purely physical measure of the shaking of the earthquake at a given location. Following Zubizarreta et al. (2013a), we used the PGA values provided by the United States Geological Survey (USGS 2011) to estimate the PGA in each of the counties where we have data. Using these values, we created three measures of exposure to the earthquake: one with three levels of shaking, another with five levels, and a final one with ten, basically defined by quantiles of exposure (see Appendix A in the Supplementary Materials for details).

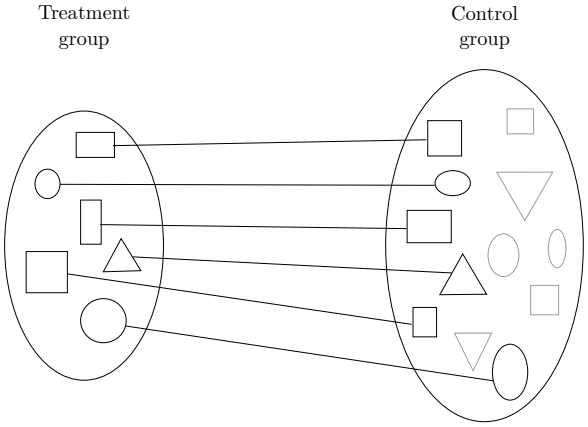
## 3 Representative matching with multi-valued treatments

As noted in Section 1.3, minimum distance matching (arguably, the mainstream form of optimal matching) with two treatments is computationally tractable (i.e., polynomially solvable), but it is computationally intractable (NP-hard) for three or more treatments. To circumvent this limitation, instead of explicitly matching samples across treatments, we will separately match each treatment sample to a representative or template sample of a target population of special interest, extending the ideas of Silber et al. (2014). This will allow us to address the two aforementioned difficulties of handling multi-valued treatments and building representative samples in matching. In the following section, we incorporate these ideas into a MIP-based matching approach for large data sets. The basic approach is depicted in Figure 1.

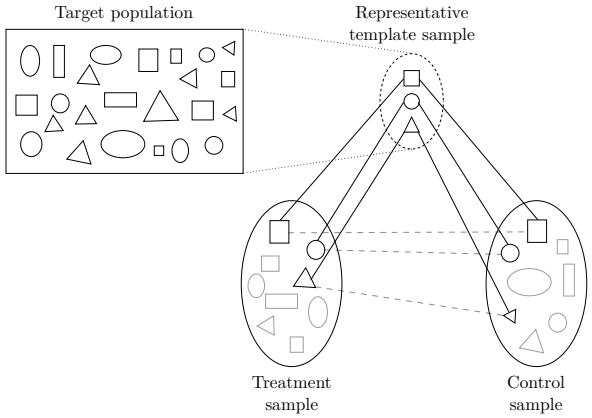
Figure 1(a) exemplifies traditional bipartite matching with a smaller treatment group than a control group. The goal is to match all the treated units to control units with similar observed covariates, represented in the figure by the shapes of the elements. Here, the target of inference is the average treatment effect on the treated. But how to target a different

Figure 1: Template matching for multi-valued treatments.

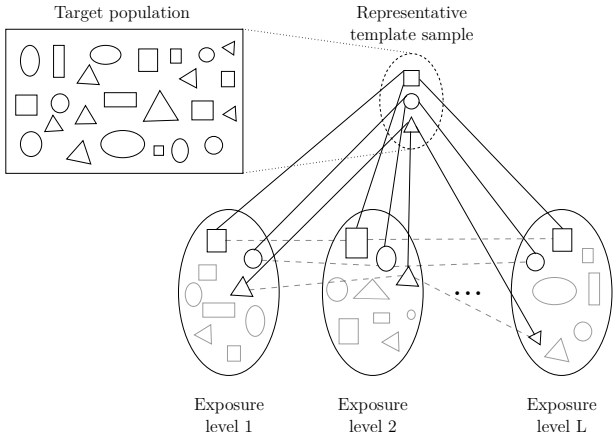
(a) Traditional matching




(b) Template matching for treatment and control groups




(c) Template matching for multi-valued (ordered or unordered) treatments, e.g. levels of exposure



estimand, one for a particular population of policy or scientific interest? Call this estimand the target average treatment effect (Kern et al. 2016). Figure 1(b) illustrates how to extend template matching for this purpose. Here, a representative template sample is selected from the target population of interest and then matched to the treatment and control groups separately. By construction, the matched groups will be balanced relative to the template sample and therefore to each other. We can repeat this process with multi-valued (ordered or un-ordered) treatments, possibly tens or hundreds treatments — as many as the data allows us to balance, as illustrated in Figure 1(c).

We select the template sample by drawing random samples of a given size from the population of interest. In our case study, the population of interest was the population of all high school students in tenth grade in 2008 in Chile, and we drew 500 random samples of size 1000 from this population. We chose this sample size due to practical restrictions and power considerations, in order to find a finely balanced matched sample for all covariates across all treatment groups and have adequate power. The template sample size is restricted by the size of the smallest treatment group. See Silber et al. (2014) for a power calculation in template matching. In our case study, we matched with a constant $1:1$ matching ratio in order to produce pair matches and have a simpler and more interpretable study design; however, it is possible to extend our approach to match with a variable matching ratio and improve power. Using, for example, the method by Zubizarreta (2015), it is also possible to relax our matching approach into a weighting approach, with weights that are not required to be constant (and therefore that will not produce a self-weighted sample), and further improve power, yet under a different mode of inference. In our case study, we follow Rosenbaum (2002) and conduct randomization based inferences.

From the 500 random samples, we selected the sample that was closest to the population in terms of a robust version of the Mahalanobis distance (Rosenbaum 2010, Chapter 8). Table 6 in the Supplementary Materials describes the population and the selected template sample in terms of the means for each category of the observed covariates. In our case study, we used this template to match representative samples for each level of exposure to the earthquake as follows.

As described in Table 6, we have 14 categorical covariates with 81 categories in total, or 81 binary covariates for each of these categories. These covariates are: the student's gender (2 categories), and ethnicity (3 categories); the father and mother's education (5 categories each), household income (7 categories), and number of books at home (6 categories); student's school attendance (10 categories), GPA ranking within the school (10 categories), and SIMCE score (11 categories); the school's type (public, voucher, or private), location (rural or urban), whether the school is catholic or not, its socioeconomic status (SES; 5 categories), and the average SIMCE score of the school (10 categories). We perfectly balanced the marginal distributions of all these covariates by matching with fine balance (Rosenbaum et al. 2007). Fine balance is a covariate balance requirement that results in perfect balance of the marginal distributions of the observed covariates; however, without matching units within the same categories of the covariates as in exact matching (see Visconti and Zubizarreta 2018 for an illustration). By matching each treatment sample with fine balance relative to the template sample, we can guarantee that the all the matched samples will be perfectly balanced relative to each other and to the template of the target population in terms of the observed covariates (see Table 3 below).

In our case study, identification of the target average treatment effect relies on strong ignorability (Rosenbaum and Rubin 1983; Imbens and Rubin 2015) of both the treatment assignment and the sample selection mechanisms (see, e.g., Tipton 2013 and Kern et al. 2016). The fact that the entire territory of the country is prone to earthquakes that people cannot anticipate, the broad scope of the covariates in our data set for which we adjust by matching, and the act of randomly selecting the template sample from the (entire) population, make these assumptions plausible. We also assume the stable unit treatment value assumption (SUTVA; Rubin 1980, 1986) holds.

# 4 Effective formulations for matching

In this section, we analyze and contrast the effectiveness of linear- and quadratic-sized mixed integer programming matching formulations. In particular, we show that a particular linear-sized formulation is more effective as it allow us to handle considerably larger data sets in a

short time. For example, in our case study, one of the data sets involved more than 70,000 units, yet the linear-sized formulation could find the solution in less than one minute using a standard laptop computer. With quadratic-sized formulations, the problem did not even fit in memory. Coupled with the matching strategy in the above section, this allows us to build balanced and representative matched samples with multi-valued treatments in considerably larger data sets.

Parts of this section are technical and so we begin by describing its structure and main results. In Section 4.1 we describe the notation and a quadratic-sized formulation for distributional balance that works well in some instances, but that cannot handle the data in our case study due to its size. In Section 4.2 we describe how the quadratic-sized formulation can be reduced to a linear-sized formulation through a simple procedure that could potentially result in a loss of formulation strength. In Section 4.3 we show that no strength is actually lost with this reduction and prove three results: the optimal objective values of the linear programming (LP) relaxations of the linear-sized and quadratic-sized formulations are identical; with two covariates or nested covariates, the LP relaxation of the linear-sized formulation is integral; and with three or more covariates, the LP relaxation of both formulations can fail to be integral. In practice, however, the smaller formulation is quite practical. As we show in Section 4.4, this formulation can handle data sets with more than 700,000 units in only a couple of minutes.

## 4.1 A large formulation for distributional balance

Let $\mathcal{T} = \{t_1, \ldots, t_T\}$ represent the $T$ units in the template sample. Put $\mathcal{L} = \{\ell_1, \ldots, \ell_L\}$ for the set of $L$ units under a given treatment or exposure level. Define $\mathbb{L}$ as the family containing all such sets $\mathcal{L}$ of units for each treatment or exposure level (we do not index $\mathbb{L}$ to emphasize that exposure levels can be unordered). Without loss of generality, fix $\mathcal{L} \in \mathbb{L}$ because, as explained in Section 3, the matching process is identical for every exposure level $\mathcal{L} \in \mathbb{L}$ (see Figure 1(c)). In addition, let $\mathcal{P} = \{p_1, \ldots, p_P\}$ denote the $P$ observed covariates that we aim to balance and let $\mathbf{x}_i = (x_{i,p})_{p \in \mathcal{P}}$ be the covariate values for the units in the template and exposure level samples for $i \in \mathcal{T}$ and $i \in \mathcal{L}$, respectively. Finally, let $\mathcal{K}(p) =$

$\{k_1, \ldots, k_{K_p}\}$ stand for the categories of covariate $p \in \mathcal{P}$, and $\mathcal{T}_{p,k} = \{t \in \mathcal{T} : x_{t,p} = k\}$ and $\mathcal{L}_{p,k} = \{\ell \in \mathcal{L} : x_{\ell,p} = k\}$ be respectively the template and level $l$ units in category $k$ in covariate $p$, with $N_{p,k} = |\mathcal{T}_{p,k}|$.

We can minimize the imbalances in the marginal distributions of the $P$ covariates, by solving the following optimization problem

$$\underset{\boldsymbol{v},\, \boldsymbol{m}}{\text{minimize}} \quad \sum\nolimits_{p\in\mathcal{P}} \sum\nolimits_{k\in\mathcal{K}(p)} v_{p,k} \tag{1a}$$

$$\text{subject to} \quad \left| \sum\nolimits_{\ell\in\mathcal{L}_{p,k}} \sum\nolimits_{t\in\mathcal{T}} m_{t,\ell} - N_{p,k} \right| \leq v_{p,k}, \qquad \forall p \in \mathcal{P}, \quad k \in \mathcal{K}(p), \tag{1b}$$

$$\sum_{t\in\mathcal{T}} m_{t,\ell} \leq 1, \qquad \forall \ell \in \mathcal{L}, \tag{1c}$$

$$\sum_{\ell\in\mathcal{L}} m_{t,\ell} = 1, \qquad \forall t \in \mathcal{T}, \tag{1d}$$

$$m_{t,\ell} \in \{0, 1\}, \quad \forall t \in \mathcal{T}, \quad \ell \in \mathcal{L}, \tag{1e}$$

where $v_{p,k}$ defines the imbalance or violations of fine balance (Rosenbaum et al. 2007) for category $k$ of covariate $p$, and $m_{t,\ell}$ is the binary decision variable which takes the value of 1 if template unit $t$ is matched to level $\mathcal{L}$ unit $\ell$, and 0 otherwise. Constraints (1c)–(1e) ensure that every unit in the template sample is matched to exactly one unit in the sample under exposure level $\mathcal{L}$ without replacement. Under these constraints, we have that $\sum_{\ell\in\mathcal{L}_{p,k}} \sum_{t\in\mathcal{T}} m_{t,\ell}$ in Constraint (1b) is equal to the total number of units in the sample under exposure level $\mathcal{L}$ with category $k$ in covariate $p$, for all $k \in \mathcal{K}(p)$ and $p \in \mathcal{P}$. Hence, for each $p \in \mathcal{P}$ and $k \in \mathcal{K}(p)$, Constraint (1b) forces $v_{p,k}$ to be greater or equal to the absolute value violation from the target $N_{p,k}$ of units with category $k$ in covariate $p$ in the template sample. Then, the objective function $\sum_{p\in\mathcal{P}} \sum_{k\in\mathcal{K}(p)} v_{p,k}$ is greater or equal to the total imbalances in the marginal distributions of the $P$ covariates. Because this objective function is being minimized, we will further have that all inequalities in (1b) will hold at equality for the optimal solution and thus that the objective value at the optimum will be equal to the total sum of covariate imbalances (otherwise we would be able to reduce the variables $v_{p,k}$ for the inequalities (1b) that hold at strict inequality, which would reduce the objective and contradict optimality).

Constraints (1c)–(1d) are linear inequalities and, for each $p \in \mathcal{P}$ and $k \in \mathcal{K}(p)$, Constraint (1b) can be replaced by the two linear inequalities

$$\sum\nolimits_{\ell \in \mathcal{L}_{p,k}} \sum\nolimits_{t \in \mathcal{T}} m_{t,\ell} - N_{p,k} \leq v_{p,k}, \tag{2a}$$

$$N_{p,k} - \sum\nolimits_{\ell \in \mathcal{L}_{p,k}} \sum\nolimits_{t \in \mathcal{T}} m_{t,\ell} \leq v_{p,k}. \tag{2b}$$

In addition, for each $t \in \mathcal{T}$ and $\ell \in \mathcal{L}$, Constraint (1e) can be replaced by $0 \leq m_{t,\ell} \leq 1$ and $m_{t,\ell} \in \mathbb{Z}$ where $\mathbb{Z}$ is the set of all integers. Then (1) is in the class of optimization problems known as *mixed integer linear programming problems* because its objective is linear and all its constraints are linear inequalities or integer constraints on some variables. Here, the term mixed is intended to note that some variables (such as the variables $v_{p,k}$) may not be required to be integer. Given that we only consider linear constraints we will simply refer to such problems as MIP formulations. We use the term *formulation* instead of *problem* to reinforce the fact that there may be different MIP formulations that are equivalent as optimization problems (see the discussion after Formulation (3) in Section 4.2). Finally, we will often make use of the variant of the MIP formulation obtained by eliminating all integrality constraints (e.g., for Formulation (1) we will replace (1e) by $m_{t,\ell} \in [0, 1]$ for all $t \in \mathcal{T}$, and $\ell \in \mathcal{L}$). We refer to this variant as the *linear programming (LP) relaxation* of the formulation because it relaxes its constraints to yield a linear programming problem (i.e., an optimization problem with a linear objective and only linear inequalities).

Formulation (1) has $T \times L + \sum_{p \in \mathcal{P}} K_p$ variables and $T + L + \sum_{p \in \mathcal{P}} K_p$ constraints excluding the variable bounds. While this problem size is polynomial on the number of units, the quadratic term $T \times L$ can be prohibitive in practice. For instance, in our case study one of the data sets results in $1.8 \times 10^6$ decision variables. This makes the problem very hard in practice and attempting to solve it with a state-of-the-art solver, Gurobi v7.5.1 on a Core i7-3770 3.40GHz workstation with 32GB of RAM, results in an out-of-memory error. We get the same error even if we attempt to solve its LP relaxation.

## 4.2 Reducing the size of the formulation

One approach to reduce the size of Formulation (1) is to define variables $z_\ell = \sum_{t\in\mathcal{T}} m_{t,\ell}$ for each $\ell \in \mathcal{L}$ and use these new variables to eliminate the $m_{t,\ell}$ variables from formulation (1). For this, we first sum constraints (1d) to obtain

$$\underset{\boldsymbol{v},\,\boldsymbol{m}}{\text{minimize}} \quad \sum\nolimits_{p\in\mathcal{P}} \sum\nolimits_{k\in\mathcal{K}(p)} v_{p,k} \tag{3a}$$

$$\text{subject to} \quad \left|\sum\nolimits_{\ell\in\mathcal{L}_{p,k}} \sum\nolimits_{t\in\mathcal{T}} m_{t,\ell} - N_{p,k}\right| \le v_{p,k}, \qquad \forall p \in \mathcal{P}, \quad k \in \mathcal{K}(p) \tag{3b}$$

$$\sum_{t\in\mathcal{T}} m_{t,\ell} \le 1, \qquad \forall \ell \in \mathcal{L} \tag{3c}$$

$$\sum_{t\in\mathcal{T}} \sum_{\ell\in\mathcal{L}} m_{t,\ell} = T, \tag{3d}$$

$$m_{t,\ell} \in \{0,1\}, \quad \forall t \in \mathcal{T}, \quad \ell \in \mathcal{L}. \tag{3e}$$

We claim that formulations (1) and (3) are equivalent in the sense that for every feasible solution for one optimization problem, there exists a feasible solution for the other problem that has the same objective value.

To see the equivalency between (1) and (3), first note that any solution to (1) is a solution to (3). Second, suppose that a solution $(\boldsymbol{v}^*,\ \boldsymbol{m}^*)$ to (3) has some $t' \in \mathcal{T}$ such that $\sum_{\ell\in\mathcal{L}} m^*_{t',\ell} \ge 2$. Then, because of (3d), there is some $t'' \in \mathcal{T}$ with $t'' \ne t'$ such that $\sum_{\ell\in\mathcal{L}} m^*_{t'',\ell} = 0$. If $\ell' \in \mathcal{L}$ is such that $m^*_{t',\ell'} = 1$ we can change the solution by letting $m^*_{t',\ell'} = 0$ and $m^*_{t'',\ell'} = 1$. This does not change $\sum_{t\in\mathcal{T}} m^*_{t,\ell}$ for any $l \in \mathcal{L}$ so $\sum_{\ell\in\mathcal{L}_{p,k}} \sum_{t\in\mathcal{T}} m^*_{t,\ell} - N_{p,k}$ also remains unchanged for all $k \in \mathcal{K}(p)$ and $p \in \mathcal{P}$. Then the modified $\boldsymbol{m}^*$ together with the original $\boldsymbol{v}^*$ remain feasible for (3). If we repeat this step, we eventually get a solution to (1) with the same objective value as the original solution.

Noting that the order of the sums in (3d) can be exchanged, we can replace every occurrence

of $\sum_{t\in\mathcal{T}} m_{t,\ell}$ by $z_\ell$ in (3) to obtain the following formulation

$$\underset{\boldsymbol{v},\,\boldsymbol{z}}{\text{minimize}} \quad \sum\nolimits_{p\in\mathcal{P}} \sum\nolimits_{k\in\mathcal{K}(p)} v_{p,k} \tag{4a}$$

$$\text{subject to} \quad \left|\sum\nolimits_{\ell\in\mathcal{L}_{p,k}} z_\ell - N_{p,k}\right| \le v_{p,k} \qquad \forall p\in\mathcal{P}, \quad k\in\mathcal{K}(p) \tag{4b}$$

$$\sum\nolimits_{\ell\in\mathcal{L}} z_\ell = T \qquad \forall p\in\mathcal{P}, \quad k\in\mathcal{K}(p) \tag{4c}$$

$$z_\ell \in \{0,1\} \quad \ell\in\mathcal{L}, \tag{4d}$$

where $z_\ell$ is a binary decision variable that takes the value 1 if unit $\ell$ of the exposure level group $\mathcal{L}$ is matched to the template $\mathcal{T}$, and 0 otherwise. We claim that Formulation (4) is equivalent to Formulation (3) because of the identity $z_\ell = \sum_{t\in\mathcal{T}} m_{t,\ell}$. In fact, given a solution $(\boldsymbol{v}^*,\ \boldsymbol{m}^*)$ to (3) we can obtain a solution $(\boldsymbol{v}^*,\ \boldsymbol{z}^*)$ to (4) by defining $\boldsymbol{z}^*$ through this identity. Conversely, given a solution $(\boldsymbol{v}^*,\ \boldsymbol{z}^*)$ for (4) we can obtain a solution $(\boldsymbol{v}^*,\ \boldsymbol{m}^*)$ to (3) by defining $\boldsymbol{m}^*$ as an arbitrary matching between the $|\mathcal{T}|$ template units and the $|\mathcal{T}|$ level $\mathcal{L}$ units for which $z_\ell = 1$.

Formulation (4) has $L + \sum_{p\in\mathcal{P}} K_p$ variables and $1 + \sum_{p\in\mathcal{P}} K_p$ constraints excluding the variable bounds, which is significantly smaller than Formulation (1). Indeed, for the aforementioned data set in our case study, going from Formulation (1) to (4) yields a reduction in the number of variables from over $1.8 \times 10^6$ to 19208 and a reduction in the number of constraints from 19370 to 162. Thanks to this drastic decrease in size we can solve the LP relaxation of (4) for this data set in less than one second.

However, this reduction in problem size and solution time for the LP relaxation may not necessarily translate into a reduction in solution time for the MIP Formulation (4). To analyze this solution time, it is convenient to consider the tractability of minimum distance matching (e.g., Rosenbaum 1989) from an LP perspective instead of the traditional graph theoretical one (i.e., the assignment problem or the bipartite matching one). From this perspective, the tractability of minimum distance matching can be attributed to the fact that the LP relaxation of its standard assignment problem formulation has extreme points or basic feasible solutions that always satisfy the integrality constraints of the formulation.

A formulation with this property is usually denoted *integral* and this property implies that solving the formulation is equivalent to solving its LP relaxation and hence minimum distance matching is polynomially solvable (Schrijver 2003). In contrast, the extreme points of the LP relaxations of (1) and (4) do not necessarily satisfy the integrality constraints. Hence, these problems are not necessarily polynomially solvable and may need to be solved with a MIP solver instead of just an LP solver. In such cases, a relatively good predictor of the difficulty of solving the optimization problem with a MIP solver is the distance between the optimal value of the original MIP problem and its LP relaxation, which is often denoted the *integrality* or *LP GAP* (Vielma 2015). If the GAP is small, the formulation is *strong* and it is expected to lead to faster solution times than similar sized *weaker* formulations with a larger GAP. With regards to LP GAP, we have that two equivalent MIP formulations are equally strong if in addition their LP relaxations are equivalent. Unfortunately, when considering equivalent MIP formulations like (1) and (4), the smaller formulation is often weaker than the larger one. In other words, equivalency of MIP formulations does not automatically imply equivalency of their LP relaxations. Nonetheless, in the following section we show that the LP relaxations of (1) and (4) are indeed equivalent and hence the improvements from the size reduction from (1) to (4) are not affected in terms of strength.

## 4.3 Complexity and strength of the formulation

Using results from Balas and Pulleyblank (1983) we can fortunately show that no formulation strength is lost in the transformation from (1) to (4). We formalize this fact in the following simple proposition for which we give a self-contained proof.

**Proposition 4.1.** *The LP relaxations of formulations* (1) *and* (4) *are equivalent. In particular,* (1) *and* (4) *have the same LP GAP. Furthermore,* (4) *is integral whenever* (1) *is integral.*

*Proof.* We follow a similar logic to our arguments for the equivalence between (1) and (3), but using continuous rather than binary variables.

First, let $(\boldsymbol{v},\ \boldsymbol{m})$ be a feasible solution to the LP relaxation of (1). Then $(\boldsymbol{v},\ \boldsymbol{z})$ for $\boldsymbol{z}$ given by $z_\ell = \sum_{t\in\mathcal{T}} m_{t,\ell}$ for all $\ell \in \mathcal{L}$ is a feasible solution to the LP relaxation of (4) obtained by

relaxing (4d) into $z_\ell \in [0,1]$.

For the converse, let $(\boldsymbol{v},\ \boldsymbol{z})$ be a feasible solution to the LP relaxation of (4). We obtain a feasible solution $(\boldsymbol{v},\ \boldsymbol{m})$ to the LP relaxation of (1) by constructing $\boldsymbol{m}$ as follows. Let $\underline{s}_1 = 0$, $\overline{s}_1 = \min\left\{s \geq 1 : \sum_{i=\underline{s}_1+1}^{s} z_{\ell_i} \geq 1\right\}$, $m_{t_1,\ell_{\overline{s}_1}} = 1 - \sum_{i=\underline{s}_1+1}^{\overline{s}_1-1} z_{\ell_i}$ and for each $i \in \{\underline{s}_1+1,\ldots,\overline{s}_1-1\}$ let $m_{t_1,\ell_i} = z_{\ell_i}$. Then, for each $j \in \{2,\ldots,T\}$ let $\underline{s}_j = \overline{s}_{j-1}$, $\overline{s}_j = \min\left\{s \geq \underline{s}_j+1 : \left(z_{\ell_{\underline{s}_j}} - m_{t_{j-1},\ell_{\underline{s}_j}}\right) + \sum_{i=\underline{s}_j+1}^{s} z_{\ell_i} \geq 1\right\}$, $m_{t_j,\ell_{\overline{s}_j}} = 1 - \sum_{i=\underline{s}_j+1}^{\overline{s}_j-1} z_{\ell_i} - \left(z_{\ell_{\underline{s}_j}} - m_{t_{j-1},\ell_{\underline{s}_j}}\right)$, $m_{t_j,\ell_{\underline{s}_j}} = \left(z_{\ell_{\underline{s}_j}} - m_{t_{j-1},\ell_{\underline{s}_j}}\right)$, and for each $i \in \left\{\underline{s}_j+1,\ldots,\overline{s}_j-1\right\}$ let $m_{t_j,\ell_i} = z_{\ell_i}$. Because of (4c) we have that $m_{t_T,\ell_{\overline{s}_T}} = z_{\ell_{\overline{s}_T}}$ and $z_{\ell_i} = 0$ for all $i > \overline{s}_T$. Then, by construction $m$ satisfies (1c)–(1d), $m_{t,\ell} \in [0,1]$ for all $t \in \mathcal{T}$ and $\ell \in \mathcal{L}$, and $z_\ell = \sum_{t\in\mathcal{T}} m_{t,\ell}$ for all $\ell \in \mathcal{L}$. Finally, because of this last equation and the fact that $(z,v)$ satisfies (4b) we have that $(\boldsymbol{v},\ \boldsymbol{z})$ satisfies (1b).

For the last statement on integrality note that we have just shown that the feasible region of the LP relaxations of (4) is the image of the feasible region of the LP relaxations of (1) through the linear transformation that preserves $\boldsymbol{v}$ and sets $z_\ell = \sum_{t\in\mathcal{T}} m_{t,\ell}$ for all $\ell \in \mathcal{L}$. Hence, all extreme points of the LP relaxations of (1) are images of the extreme points of the LP relaxations of (4) through this same linear transformation. The result follows because this linear transformation sends integral $\boldsymbol{m}$ to integral $\boldsymbol{z}$. □

Proposition 4.1 shows that we do not lose formulation strength with the size reduction, but it does not tell us how strong Formulation (4) is. The following proposition, which we prove in Appendix C in the Supplementary Materials, shows that (4) is integral for problems with at most two covariates or with the nested covariate structure considered by Pimentel et al. (2015). As an example of this nested structure, consider three covariates, with all the categories of the third covariate being subcategories of the second covariate, and these categories in turn being subcategories of the first covariate; therefore, the covariates categories are progressively finer and nested from the first to the third covariate. In such cases, we have that, similar to the assignment formulations for propensity score matching, the solution of (4) is equivalent to the solution of its LP relaxations.

**Proposition 4.2.** *The LP relaxation of* (4) *is integral if*

1. $P \leq 2$, *or*

2. *for all* $i < j$ *and* $k \in \mathcal{K}(p_j)$ *there exist* $k' \in \mathcal{K}(p_i)$ *such that* $\mathcal{L}_{p_j,k} \subseteq \mathcal{L}_{p_i,k'}$ *and* $\mathcal{T}_{p_j,k} \subseteq \mathcal{T}_{p_i,k'}$.

*Hence, under these conditions* (4) *can be solved in polynomial time by solving its LP relaxation.*

Unfortunately, the following lemma, that we prove in Appendix B, shows that both formulations can fail to be integral for non-nested covariates, even if $P = 3$.

**Lemma 4.3.** *Even for* $P = 3$ *and* $K_1 = K_2 = K_3 = 3$*, there exist covariates for which the LP relaxations of* (1) *and* (4) *fail to be integral.*

This loss of integrality for more general covariate structures is not surprising as the problem is NP-hard. However, as we show in Section 4.4, Formulation (4) remains computationally effective even when the covariate assumptions of Proposition 4.2 are not satisfied, and we can still solve all the matching problems in our case study in a few seconds using a regular laptop (see Table 2). These computational results suggest that enough of the formulation strength is preserved even when Proposition 4.2 does not hold (Vielma 2015).

## 4.4 Performance of the linear-sized formulation

To evaluate the above ideas and results in practice, we implemented them for cardinality matching (Zubizarreta et al. 2014) in the new function `cardmatch` in the package `designmatch` for `R` (Zubizarreta et al. 2018). We tested this function in data sets of different sizes. We increased the largest exposure sample in our data set ($L = 70118$) up to ten times ($L = 701180$) by creating random copies of it. For each copy, we randomly modified all the nominal covariates by adding 1, 0, or -1, and truncating the resulting values to preserve the original ranges. We also created templates of different sizes of up to 10000 observations.

Table 1 shows the computing times. The largest data set we considered had a template sample size of $T = 10000$ and an exposure sample size of $L = 701180$, and took approximately three minutes. Most of the matches took less than two minutes. Naturally, the computing time tends to increase both with the template and exposure sample sizes.

Table 1: Computing time (in minutes) of the proposed matching formulation as implemented in `designmatch` for `R` (Zubizarreta et al. 2018). We considered different template samples sizes (ranging from 1000 to 10000) and varying exposure sample sizes, increasing from the largest exposure sample size in our case study, 70118, to $10 \times 70118 = 701180$.

| | Exposure size $L$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Template size $T$ | 70118 | 140236 | 210354 | 280472 | 350590 | 420708 | 490826 | 560944 | 631062 | 701180 |
| 1000 | 0.28 | 0.50 | 0.65 | 0.79 | 1.11 | 1.20 | 1.49 | 2.13 | 2.58 | 2.63 |
| 2000 | 0.20 | 0.72 | 0.91 | 1.14 | 1.49 | 1.56 | 1.87 | 2.20 | 2.53 | 2.67 |
| 3000 | 0.19 | 0.73 | 1.08 | 1.37 | 1.62 | 1.51 | 2.02 | 2.26 | 2.53 | 3.15 |
| 4000 | 0.22 | 0.44 | 1.09 | 1.57 | 1.74 | 1.98 | 2.00 | 2.29 | 2.48 | 2.62 |
| 5000 | 0.18 | 0.33 | 0.87 | 1.26 | 1.52 | 1.94 | 3.05 | 1.73 | 2.93 | 3.51 |
| 6000 | 0.26 | 0.47 | 0.64 | 1.66 | 2.07 | 2.40 | 2.78 | 2.94 | 3.18 | 3.04 |
| 7000 | 0.18 | 0.36 | 0.56 | 0.76 | 1.62 | 2.09 | 2.28 | 2.36 | 2.71 | 8.54 |
| 8000 | 0.25 | 0.40 | 0.57 | 0.82 | 1.87 | 2.25 | 2.42 | 2.95 | 3.08 | 3.66 |
| 9000 | 0.25 | 0.46 | 0.74 | 0.82 | 0.99 | 2.18 | 2.94 | 3.13 | 4.13 | 3.85 |
| 10000 | 0.19 | 0.39 | 0.63 | 0.83 | 1.08 | 2.55 | 2.58 | 2.93 | 3.13 | 3.42 |

Table 2 shows the matching time in our case study with 3, 5, and 10 levels of exposure, and a template sample size of $T = 1000$. The table shows that all matching times are well under a minute, and most of them are under 10 seconds. The computing times do not increase monotonically with the sample size in part due to random variation in the generation of the covariate values. We also compared the performance of the linear-sized formulation in `cardmatch` to other matching packages based on quadratic-formulations, such as `optmatch` (Hansen 2007) and `rcbalance` (Pimentel et al. 2015). Trying to run `optmatch` results in an out-of-memory error, even for our original case study sample ($T = 1000$ and $L = 70118$). In the case of `rcbalance`, we were only able to match the original sample ($L = 70118$) to template sizes up to size $T = 6000$. For larger template sizes, we were not able to get a solution in the allowed time of 8 hours.

Table 2: Computing time (in minutes) in the case study with 3, 5, and 10 levels of exposure to the earthquake.

| Exposure level | Sample size | Time (min) |
|---|---|---|
| 1 | 18208 | 0.05 |
| 2 | 70118 | 0.34 |
| 3 | 32953 | 0.08 |
| 1 | 22075 | 0.06 |
| 2 | 25977 | 0.06 |
| 3 | 24896 | 0.06 |
| 4 | 24279 | 0.06 |
| 5 | 24052 | 0.06 |
| 1 | 12084 | 0.03 |
| 2 | 9991 | 0.03 |
| 3 | 12513 | 0.04 |
| 4 | 13464 | 0.03 |
| 5 | 13119 | 0.03 |
| 6 | 11777 | 0.04 |
| 7 | 12813 | 0.03 |
| 8 | 11466 | 0.03 |
| 9 | 12071 | 0.03 |
| 10 | 11981 | 0.03 |

## 4.5 Further considerations

While our matching approach, as described in equations (4a)–(4d), is applicable to categorical covariates, it can be extended to balance continuous covariates too. To see this, let $x_q$ be a continuous covariate and $f(x_q)$ be a suitable transformation of $x_q$. Define $f^*(x_q)$ as the value of $f(x_q)$ in the target (e.g., as the empirical value of $f(x_q)$ in the template sample or ideally in the original target population itself). Replace constraint (4b) for $\left|\sum_{\ell\in\mathcal{L}} \frac{f(x_{\ell,q})z_\ell}{T} - f^*(x_q)\right| \leq v_q$, where $x_{\ell,q}$ is the observed value of covariate $x_q$ for unit $\ell \in \mathcal{L}$. Then, the resulting formulation will minimize the imbalances between the matched sample and the target distribution as characterized by $f^*(x_q)$. For instance, if $f$ is the identity, then the resulting formulation will minimize the imbalances in the means of $x_q$. Along the same lines, if $f$ is an indicator for the quantiles of $x_q$, then it will balance a coarsened version of the Kolmogorov-Smirnov test statistics as in Zubizarreta (2012). In general, $f$ can represent basis functions on the covariates (e.g., Wang and Zubizarreta 2020) and the matching formulations in Zubizarreta (2012) can be reformulated using the approach in (4). The previous computational guarantees in Proposition 4.2 and Lemma 4.3, nonetheless, would have to be carefully studied.

We also note that, unlike the original template matching approach by Silber et al. (2014), this

approach for matching for balance does not require a target sample with individual data, but rather a target population characterized by aggregate statistics of the joint distribution of its observed covariates. This gives additional versatility to our approach. After matching for balance, the selected balanced samples can always be re-matched for homogeneity in order to increase efficiency and reduce the sensitivity to hidden biases of certain test statistics (see Zubizarreta et al. 2014 for details).

# 5 Results from the case study

## 5.1 Assessing balance

Table 3 shows covariate balance for the matchings with 3, 5, and 10 levels of exposure to the earthquake. In the template (target) sample there are 490 female and 510 male students, and the same is true across all exposure levels. Due to space constraints, the table shows the counts for 3 covariates only (gender, school SES, and mother's education), but the same pattern holds for all other 11 covariates (see Figure 3 in the Supplementary Materials). In other words, the marginal distributions of the 14 covariates are perfectly balanced relative to each other and to the representative template sample. As a result, Figure 3 shows that means of the indicators for the 81 covariate categories are perfectly balanced after matching.

Table 3: Distributional balance or fine balance across matched samples for 3, 5, and 10 levels of exposure to the earthquake. Due to space constraints, the counts are shown for three covariates only but the same pattern holds for all other 11 covariates.

(a) 3 exposure levels

| | | Exposure level | | |
|---|---|---|---|---|
| Covariate | Template | 1 | 2 | 3 |
| Gender | | | | |
| Male | 490 | 490 | 490 | 490 |
| Female | 510 | 510 | 510 | 510 |
| School SES | | | | |
| Low | 90 | 90 | 90 | 90 |
| Mid-low | 318 | 318 | 318 | 318 |
| Medium | 303 | 303 | 303 | 303 |
| Mid-high | 174 | 174 | 174 | 174 |
| High | 115 | 115 | 115 | 115 |
| Mother's education | | | | |
| Primary | 321 | 321 | 321 | 321 |
| Secondary | 434 | 434 | 434 | 434 |
| Technical | 120 | 120 | 120 | 120 |
| College | 115 | 115 | 115 | 115 |
| Missing | 10 | 10 | 10 | 10 |
| ⋮ | | | | |

(b) 5 exposure levels

| | | Exposure level | | | | |
|---|---|---|---|---|---|---|
| Covariate | Template | 1 | 2 | 3 | 4 | 5 |
| Gender | | | | | | |
| Male | 490 | 490 | 490 | 490 | 490 | 490 |
| Female | 510 | 510 | 510 | 510 | 510 | 510 |
| School SES | | | | | | |
| Low | 90 | 90 | 90 | 90 | 90 | 90 |
| Mid-low | 318 | 318 | 318 | 318 | 318 | 318 |
| Medium | 303 | 303 | 303 | 303 | 303 | 303 |
| Mid-high | 174 | 174 | 174 | 174 | 174 | 174 |
| High | 115 | 115 | 115 | 115 | 115 | 115 |
| Mother's education | | | | | | |
| Primary | 321 | 321 | 321 | 321 | 321 | 321 |
| Secondary | 434 | 434 | 434 | 434 | 434 | 434 |
| Technical | 120 | 120 | 120 | 120 | 120 | 120 |
| College | 115 | 115 | 115 | 115 | 115 | 115 |
| Missing | 10 | 10 | 10 | 10 | 10 | 10 |
| ⋮ | | | | | | |

(c) 10 exposure levels

| | | Exposure level | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Covariate | Template | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| Gender | | | | | | | | | | | |
| Male | 490 | 490 | 490 | 490 | 490 | 490 | 490 | 490 | 490 | 490 | 490 |
| Female | 510 | 510 | 510 | 510 | 510 | 510 | 510 | 510 | 510 | 510 | 510 |
| School SES | | | | | | | | | | | |
| Low | 90 | 90 | 90 | 90 | 90 | 90 | 90 | 90 | 90 | 90 | 90 |
| Mid-low | 318 | 318 | 318 | 318 | 318 | 318 | 318 | 318 | 318 | 318 | 318 |
| Medium | 303 | 303 | 303 | 303 | 303 | 303 | 303 | 303 | 303 | 303 | 303 |
| Mid-high | 174 | 174 | 174 | 174 | 174 | 174 | 174 | 174 | 174 | 174 | 174 |
| High | 115 | 115 | 115 | 115 | 115 | 115 | 115 | 115 | 115 | 115 | 115 |
| Mother's education | | | | | | | | | | | |
| Primary | 321 | 321 | 321 | 321 | 321 | 321 | 321 | 321 | 321 | 321 | 321 |
| Secondary | 434 | 434 | 434 | 434 | 434 | 434 | 434 | 434 | 434 | 434 | 434 |
| Technical | 120 | 120 | 120 | 120 | 120 | 120 | 120 | 120 | 120 | 120 | 120 |
| College | 115 | 115 | 115 | 115 | 115 | 115 | 115 | 115 | 115 | 115 | 115 |
| Missing | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 |
| ⋮ | | | | | | | | | | | |

## 5.2 Visualizing effects

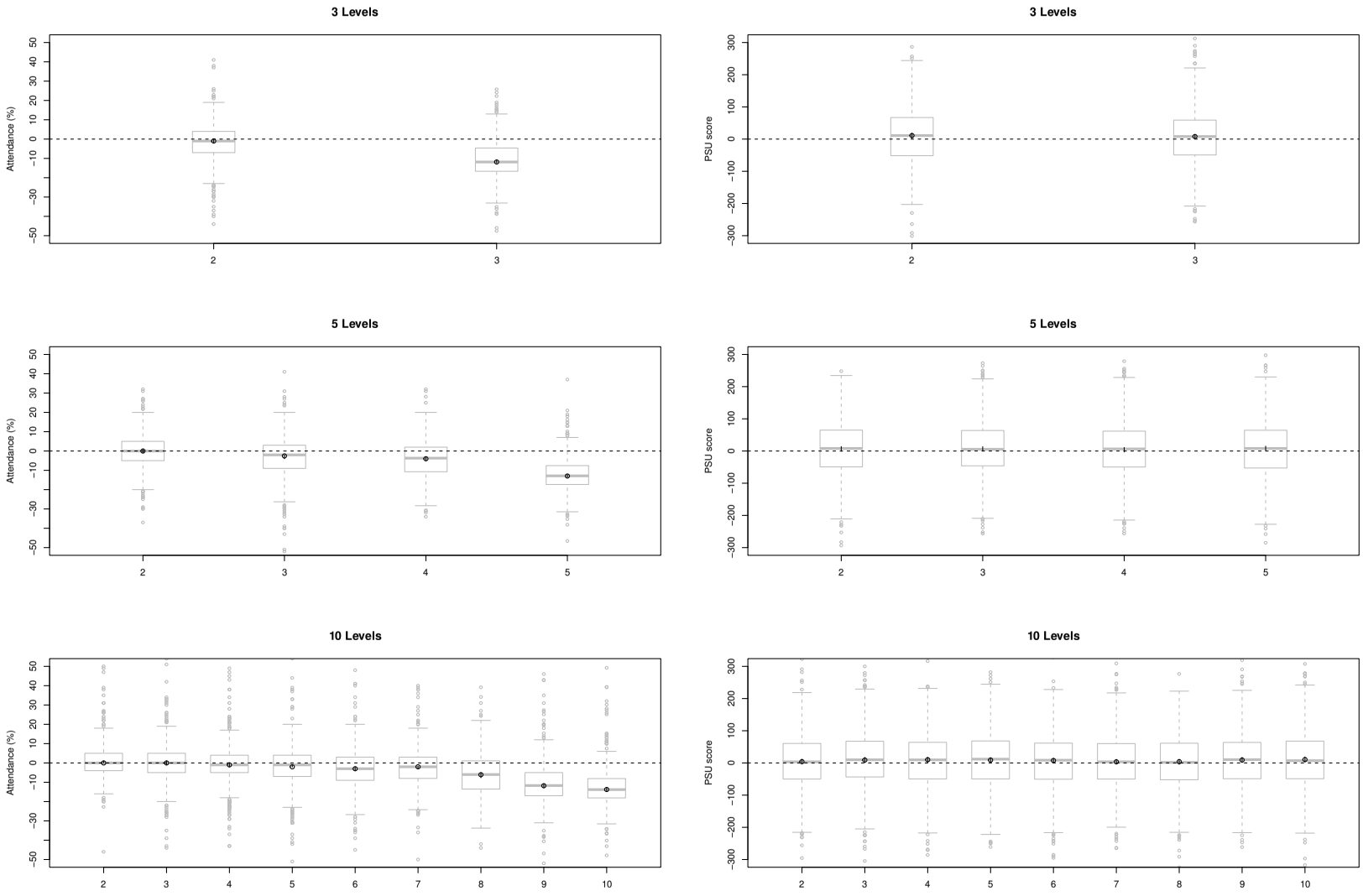
One of the advantages of matching methods is that their adjustments are transparent, as illustrated in Table 3. Also, the adjustments are made without looking at outcomes, which aids the objectivity of the observational study (Rubin 2008). Furthermore, the structure of the data after adjustments is simple enough that we can analyze the effects by simply taking differences in means and even by plotting the outcomes. This is illustrated in Figure 2.

Figure 2 shows the distribution of matched-pair differences in outcomes for both school attendance (Figure 2(a)) and PSU scores (Figure 2(b)) for each higher exposure level relative to level 1. As expected, Figure 2(a) shows that as the exposure level increases, the impact on school attendance becomes more severe. However, this pattern stands in stark contrast to the one in Figure 2(b), where we see no effect of the earthquake on university admission test scores. This is striking, given the magnitude of the earthquake and its impact on school attendance that year. Previous studies have documented a positive effect of school attendance on student achievement as measured by test scores (e.g., Lamdin 1996; Gottfried 2010; Paredes and Ugarte 2011), but this does not appear to be the case in the last year of school in Chile in terms of PSU scores. In the following subsection we estimate the actual effects.

Figure 2: Matched-pair differences in outcomes with respect to level 1

(a) Pair differences for attendance

(b) Pair differences for the PSU scores



## 5.3 Estimating contrasts

To estimate the effect of different levels of exposure to the earthquake, we compute Hodges-Lehmann point estimates as well as 95% confidence intervals following the procedures in Section 7.4 of Hollander et al. (2015). See also Shuster and Boyett (1979). We compare the outcome of interest for each exposure level $\mathcal{L} \in \mathbb{L}$ with respect to the control level $\mathcal{L}_1$. We consider the null hypothesis

$$H_0 : \ \tau_u^i = \tau_1^i \text{ if } |R_u^i - R_1^i| < r_{\alpha/2}^{i*},$$

where $\tau_j^i$ is the treatment effect for level $j$ in $\mathcal{L}$, $R_j^i$ is the sum of the within-matched group ranks, and $r_{\alpha/2}^{i*}$ is a scalar such that

$$\Pr_0(|R_u^i - R_1^i| < r_{\alpha/2}^{i*}, u = 2, ..., L) = 1 - \alpha.$$

By setting an experiment-wise error rate of $\alpha$, we address the issue of multiple comparisons across different exposure levels. In practice, we obtained the value of $r_{\alpha/2}^{i*}$ using the function `cNWWM` in `NSM3` for R (Schneider et al. 2016).

Table 7 in Appendix E in the Supplementary Materials shows the effect estimates across exposure levels. The results are consistent with the boxplots in Figure 2. For example, in Table 7(c) with ten levels of exposure, the impact of the earthquake on school attendance increases with the exposure level, having an effect of 3 percentage points for level 6 and of almost 14 points for level 10, both relative to exposure level 1. For exposure levels lower than 4, the effects on school attendance are not statistically significant. In the second column of the table, it is again surprising to see that despite this negative and significant impact on school attendance, the earthquake did not have a significant negative impact on the PSU scores.

To assess the sensitivity of these findings to hidden biases, we compute Rosenbaum bounds (1987b; 2002; 2015) adjusting for multiple comparisons with Holm's procedure (Fogarty and Small 2016). These bounds quantify the magnitude of effect that an unobserved covariate

would need to have in order to materially alter the conclusions of the study. In tables 4 and 5, this magnitude is summarized by the parameter $\Gamma$ for 3, 5, and 10 levels of exposures.[2]

For our two outcomes (school attendance and PSU scores), we perform two separate sensitivity analyses. In Table 4, for school attendance (for which we obtained statistically significant effect estimates), $\Gamma$ quantifies the magnitude of effect that an unobserved covariate would need to have in order to explain away the significant effect estimates and render them insignificant. In Table 4, values of $\Gamma$ equal to 1 correspond to effect estimates that are insignificant in the absence of an unobserved covariate. In Table 5, for the PSU scores (for which we did not obtain statistically significant effect estimates), we perform a sensitivity analysis on an equivalence test (Rosenbaum and Silber 2009), and $\Gamma$ quantifies the magnitude of effect that an unobserved covariate would need to have in order to mask effects of magnitudes of 0.1 and 0.2 standard deviations and make them appear as if they were not present. In Table 5, values of $\Gamma$ equal to 1 correspond to estimates that remain insignificant in an equivalence test for effects of the given magnitudes in standard deviations. We considered effects of magnitudes of 0.1 and 0.2 standard deviations as they are considered to be meaningful effects in the education literature (see also Zubizarreta and Keele 2017).

On the one hand, for school attendance, we see that for high levels of exposure, the effect estimates are quite insensitive to hidden biases. For example, with 10 exposure levels (Table 4(c)) in order to explain away the estimated effect of exposure level 10 relative to level 1 on school attendance, an unobserved covariate that is perfectly associated with this outcome would need to increase the odds of being exposed to level 10 as opposed to level 1 by a factor of 16. For exposure levels 9 and 8, these values are 8.5 and 3.2, respectively. By way of contrast, Hammond's (1964) classical study on the effect of smoking on lung cancer becomes sensitive at $\Gamma \approx 6$ (Rosenbaum 2002; Section 4.3.2). On the other hand, for PSU scores, we find that the results are quite sensitive to hidden bias, as for most estimates, small biases would be able to mask positive effects of 0.1 and 0.2 standard deviations.

[2]We estimate the sensitivity parameters $\Gamma$ using the `sensmv` function in the R package `sensitivitymv` (Rosenbaum 2013)

Table 4: Sensitivity analysis on school attendance

(a) 3 levels

| Exposure level | $\Gamma$ |
|---|---|
| 2 | 1.34 |
| 3 | 10.21 |

(b) 5 levels

| Exposure level | $\Gamma$ |
|---|---|
| 2 | 1.00 |
| 3 | 1.78 |
| 4 | 2.47 |
| 5 | 15.89 |

(c) 10 levels

| Exposure level | $\Gamma$ |
|---|---|
| 2 | 1.00 |
| 3 | 1.00 |
| 4 | 1.02 |
| 5 | 1.30 |
| 6 | 1.84 |
| 7 | 1.62 |
| 8 | 3.22 |
| 9 | 8.46 |
| 10 | 15.71 |

Table 5: Sensitivity analysis on PSU scores

(a) 3 levels

| Exposure level | $\Gamma$ ($\delta = 0.1$ SD) | $\Gamma$ ($\delta = 0.2$ SD) |
|---|---|---|
| 2 | 1.00 | 1.30 |
| 3 | 1.05 | 1.43 |

(b) 5 levels

| Exposure level | $\Gamma$ ($\delta = 0.1$ SD) | $\Gamma$ ($\delta = 0.2$ SD) |
|---|---|---|
| 2 | 1.00 | 1.29 |
| 3 | 1.00 | 1.27 |
| 4 | 1.08 | 1.47 |
| 5 | 1.00 | 1.23 |

(c) 10 levels

| Exposure level | $\Gamma$ ($\delta = 0.1$ SD) | $\Gamma$ ($\delta = 0.2$ SD) |
|---|---|---|
| 2 | 1.03 | 1.41 |
| 3 | 1.00 | 1.16 |
| 4 | 1.00 | 1.25 |
| 5 | 1.00 | 1.20 |
| 6 | 1.00 | 1.37 |
| 7 | 1.06 | 1.44 |
| 8 | 1.09 | 1.49 |
| 9 | 1.00 | 1.22 |
| 10 | 1.00 | 1.13 |

# 6 Summary and remarks

In this paper, we have proposed a new approach to address three challenges in matching in observational studies. The first challenge relates to handling multi-valued treatments (possibly tens or hundreds of them, either ordered or un-ordered) without explicitly estimating the generalized propensity score, directly balancing the observed covariates, and facilitating transparent analysis of the outcomes, such as graphical displays. The second challenge relates to building matched samples that are not only balanced but also representative of a population of interest, in such as way that we obtain "representative estimates" of target causal effects. Arguably, this second challenge goes beyond matching and also applies to regression as a method for covariate adjustment for causal inference (see Aronow and Samii 2016). The third challenge relates to matching in larger data sets than usually considered, with hundred of thousands of observations, a problem which is not typically feasible in a short period of time.

To overcome these challenges, instead of simultaneously matching across treatments groups, we separately match each treatment group to a representative random sample of the population of interest. For this, we impose exact distributional or fine balance constraints. This guarantees that the marginal distributions of the matched samples across treatment groups will be identical to each other and to the template in terms of the observed covariates. The effectiveness of the approach relies on the use of a linear-sized MIP formulation, which we show (i) is as strong as much larger quadratic-sized formulations, and (ii) is integral (and hence polynomially solvable) when only two covariates or several nested covariates are considered. This integrality property is not preserved for more general covariate structures, but the formulation still retains its practical effectiveness in such settings.

We have used this new matching approach to estimate the impact of an earthquake on the educational achievement of high school students. In particular, we estimated the effect of levels of exposure to the 2010 Chilean earthquake on both school attendance and university admission test scores. We documented negatively increasing effects of the strength of the earthquake on school attendance, but no effect on university admission test scores. While

this new approach to matching focuses on obtaining balanced samples, it can be followed by re-matching the units in the balanced samples preserving covariate balance but increasing efficiency and reducing sensitivity to hidden biases as in Zubizarreta et al. (2014). Other promising extensions of this matching approach include quality measurement with a flexible matching ratio (Silber et al. 2014) and observational studies with $2^K$ factorial designs (Dasgupta et al. 2015).

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# Online Supplementary Materials

## Appendix A: Earthquake intensity levels

Using the estimated PGA, we created three measures of exposure to the earthquake: one with three levels, another with five levels, and a final one with ten levels.

We defined the exposure with three intensity levels as follows.

- Low PGA (PGA$<$ 0.08): felt by many people indoors; no buildings received damage.
- Medium PGA ($0.08 \leq$ PGA $\leq 0.25$): felt by most or all people indoors; some people were frightened; damages in some (non-resistant) buildings.
- High PGA (PGA$>$ 0.25): many people were frightened; severe shaking; damages in resistant buildings.

For the versions of the exposure with five and ten levels, we divided the students into PGA quintiles and deciles, respectively.

## Appendix B: Covariate profiles

Table 6: Covariate profile of the population and template sample

| Covariate | Population | Template |
|---|---|---|
| Female | 0.54 | 0.51 |
| Indigenous | | |
| Indigenous | 0.08 | 0.07 |
| Missing | 0.15 | 0.14 |
| Father's education | | |
| Secondary | 0.39 | 0.40 |
| Technical | 0.09 | 0.09 |
| College | 0.15 | 0.14 |
| Missing | 0.05 | 0.04 |
| Mother's education | | |
| Secondary | 0.41 | 0.43 |
| Technical | 0.13 | 0.12 |
| College | 0.12 | 0.12 |
| Missing | 0.01 | 0.01 |
| Household income (2008 CL$1000) | | |
| 100-200 | 0.26 | 0.26 |
| 200-400 | 0.30 | 0.31 |
| 400-600 | 0.13 | 0.12 |
| 600-1400 | 0.13 | 0.14 |
| 1400 or more | 0.09 | 0.09 |
| Missing | 0.01 | 0.02 |
| Number of books at home | | |
| 1-10 | 0.19 | 0.19 |
| 11-50 | 0.46 | 0.46 |
| 51-10 | 0.16 | 0.16 |
| More than 100 | 0.16 | 0.16 |
| Missing | 0.01 | 0.01 |

Table 6 (continued): Covariate profile of the population and template sample

| Covariate | Population | Template |
|---|---|---|
| Student's attendance (deciles) | | |
| 2 | 0.12 | 0.11 |
| 3 | 0.08 | 0.09 |
| 4 | 0.10 | 0.10 |
| 5 | 0.13 | 0.13 |
| 6 | 0.08 | 0.08 |
| 7 | 0.09 | 0.09 |
| 8 | 0.10 | 0.09 |
| 9 | 0.11 | 0.10 |
| 10 | 0.15 | 0.16 |
| Student's GPA 2008 (deciles) | | |
| 2 | 0.11 | 0.11 |
| 3 | 0.11 | 0.09 |
| 4 | 0.10 | 0.11 |
| 5 | 0.10 | 0.10 |
| 6 | 0.10 | 0.11 |
| 7 | 0.09 | 0.09 |
| 8 | 0.10 | 0.10 |
| 9 | 0.09 | 0.10 |
| 10 | 0.08 | 0.08 |
| Student's test scores (deciles) | | |
| 2 | 0.08 | 0.07 |
| 3 | 0.09 | 0.10 |
| 4 | 0.09 | 0.10 |
| 5 | 0.10 | 0.10 |
| 6 | 0.10 | 0.11 |
| 7 | 0.11 | 0.10 |
| 8 | 0.12 | 0.12 |
| 9 | 0.12 | 0.11 |
| 10 | 0.12 | 0.13 |
| Missing | 0.01 | 0.01 |
| School administration | | |
| Public | 0.34 | 0.34 |
| Private subsidized (voucher) | 0.55 | 0.55 |
| Rural school | 0.03 | 0.02 |
| Catholic school | 0.24 | 0.23 |
| School SES | | |
| Mid-low | 0.32 | 0.32 |
| Medium | 0.29 | 0.30 |
| Mid-high | 0.18 | 0.17 |
| High | 0.11 | 0.12 |
| School's test scores (deciles) | | |
| 2 | 0.07 | 0.07 |
| 3 | 0.09 | 0.07 |
| 4 | 0.09 | 0.09 |
| 5 | 0.10 | 0.11 |
| 6 | 0.11 | 0.12 |
| 7 | 0.10 | 0.11 |
| 8 | 0.12 | 0.12 |
| 9 | 0.12 | 0.11 |
| 10 | 0.13 | 0.13 |

# Appendix C: Proofs

To prove Proposition 4.2 we use the following lemma.

**Lemma 6.1.** *Let* $(\boldsymbol{z}, \boldsymbol{v})$ *be a feasible solution for the LP relaxation of* (4) *that satisfies all inequalities* (4b) *at equality. Furthermore, let* $\underline{\boldsymbol{z}}$ *and* $\overline{\boldsymbol{z}}$ *be such that*

- $\boldsymbol{z} = \frac{1}{2}\underline{\boldsymbol{z}} + \frac{1}{2}\overline{\boldsymbol{z}}$ *and,*
- $\max\left\{\left|\sum_{\ell\in\mathcal{L}_{p,k}} \underline{z}_\ell - \sum_{\ell\in\mathcal{L}_{p,k}} z_\ell\right|, \left|\sum_{\ell\in\mathcal{L}_{p,k}} \overline{z}_\ell - \sum_{\ell\in\mathcal{L}_{p,k}} z_\ell\right|\right\} \leq v_{p,k}$ *for all* $p \in \mathcal{P}$ *and* $k \in \mathcal{K}(p)$.

*If* $\underline{\boldsymbol{v}}$ *and* $\overline{\boldsymbol{v}}$ *are such that for all* $p \in \mathcal{P}$ *and* $k \in \mathcal{K}(p)$,

- $\underline{v}_{p,k} = v_{p,k} + \text{sign}\left(\sum_{\ell\in\mathcal{L}_{p,k}} z_\ell - N_{p,k}\right)\left(\sum_{\ell\in\mathcal{L}_{p,k}} \underline{z}_\ell - \sum_{\ell\in\mathcal{L}_{p,k}} z_\ell\right)$, *and*
- $\overline{v}_{p,k} = v_{p,k} + \text{sign}\left(\sum_{\ell\in\mathcal{L}_{p,k}} z_\ell - N_{p,k}\right)\left(\sum_{\ell\in\mathcal{L}_{p,k}} \overline{z}_\ell - \sum_{\ell\in\mathcal{L}_{p,k}} z_\ell\right)$,

*where* $\text{sign}(w) = w/|w|$, *then we have that*

$$(\underline{\boldsymbol{z}}, \underline{\boldsymbol{v}}) \text{ and } (\overline{\boldsymbol{z}}, \overline{\boldsymbol{v}}) \text{ are feasible for the LP relaxation of (4), and} \tag{5a}$$

$$(\boldsymbol{z},\ \boldsymbol{v}) = \frac{1}{2}(\underline{\boldsymbol{z}}, \underline{\boldsymbol{v}}) + \frac{1}{2}(\overline{\boldsymbol{z}}, \overline{\boldsymbol{v}}). \tag{5b}$$

*Proof.* Condition (5a) follows by noting that under the equality assumption on (4b), for all $p \in \mathcal{P}$ and $k \in \mathcal{K}(p)$ we have that if $\varepsilon < v_{p,k}$, then

$$\left|\left(\sum\nolimits_{\ell\in\mathcal{L}_{p,k}} z_\ell\right) + \varepsilon - N_{p,k}\right| = v_{p,k} + \text{sign}\left(\sum\nolimits_{\ell\in\mathcal{L}_{p,k}} z_\ell - N_{p,k}\right)\varepsilon \tag{6}$$

$$\left|\left(\sum\nolimits_{\ell\in\mathcal{L}_{p,k}} z_\ell\right) - \varepsilon - N_{p,k}\right| = v_{p,k} - \text{sign}\left(\sum\nolimits_{\ell\in\mathcal{L}_{p,k}} z_\ell - N_{p,k}\right)\varepsilon. \tag{7}$$

Condition (5b) follows by noting that $\boldsymbol{z} = \frac{1}{2}\underline{\boldsymbol{z}} + \frac{1}{2}\overline{\boldsymbol{z}}$ implies $\left(\sum_{\ell\in\mathcal{L}_{p,k}} \underline{z}_\ell - \sum_{\ell\in\mathcal{L}_{p,k}} z_\ell\right) = -\left(\sum_{\ell\in\mathcal{L}_{p,k}} \overline{z}_\ell - \sum_{\ell\in\mathcal{L}_{p,k}} z_\ell\right)$ for all $p \in \mathcal{P}$ and $k \in \mathcal{K}(p)$.

□

*Proof of Proposition 4.2.* In both cases we show that for any non-integral point $(\boldsymbol{z},\ \boldsymbol{v})$ that is feasible for the LP relaxation of (4), there exist $(\underline{\boldsymbol{z}}, \underline{\boldsymbol{v}})$ and $(\overline{\boldsymbol{z}}, \overline{\boldsymbol{v}})$ that satisfy (5) and $(\underline{\boldsymbol{z}}, \underline{\boldsymbol{v}}) \neq (\overline{\boldsymbol{z}}, \overline{\boldsymbol{v}})$, which implies that non-integral points cannot be extreme points.

First note that if there exist $p \in \mathcal{P}$ and $k \in \mathcal{K}(p)$ for which $\varepsilon := v_{p,k} - \left|\sum_{\ell\in\mathcal{L}_{p,k}} z_\ell - N_{p,k}\right| > 0$, then by letting $(\underline{\boldsymbol{z}}, \underline{\boldsymbol{v}}) = (\overline{\boldsymbol{z}}, \overline{\boldsymbol{v}}) = (\boldsymbol{z},\ \boldsymbol{v})$ and changing $\underline{v}_{p,k}$ to $v_{p,k} - \varepsilon$ and $\overline{v}_{p,k}$ to $v_{p,k} + \varepsilon$ we satisfy (5) and $(\underline{\boldsymbol{z}}, \underline{\boldsymbol{v}}) \neq (\overline{\boldsymbol{z}}, \overline{\boldsymbol{v}})$. Hence, without loss of generality we may assume that $(\boldsymbol{z},\ \boldsymbol{v})$ satisfies all constraints (4b) at equality.

Under the equality assumption on (4b), for all $p \in \mathcal{P}$ and $k \in \mathcal{K}(p)$ we have that

$$v_{p,k} \notin \mathbb{Z}_+ \text{ or } |\{\ell \in \mathcal{L}_{p,k} \,:\, z_\ell \in (0,1)\}| \neq 1. \tag{8}$$

For case 1 it suffices to prove the result for $P = 2$ (The result follows for $P = 1$ by copying the covariate as the result for $P = 2$ does not require that the covariates are different). For $P = 2$, we may use (8) (and the assumption that at least one $z_l$ is fractional) to construct sequences $\{s_1, \ldots, s_S\} \subseteq \{1, \ldots, L\}$ and $\{r_1, \ldots, r_{S+1}\} \subseteq \{1, \ldots, \max\{K_{p_1}, K_{p_2}\}\}$) such that (without loss of generality by possibly interchanging $p_1$ and $p_2$):

- $s_i \neq s_j$ for all $i, j \in \{1, \ldots, S\}$ with $i \neq j$,
- $r_i \neq r_j$ for all $i, j \in \{1, \ldots, S+1\}$ with $i < j$, $(i-j)/2 \in \mathbb{Z}$ and $(i, j) \neq (1, S+1)$,
- $z_{\ell_{s_j}} \in (0, 1)$ for all $j \in \{1, \ldots, S\}$,
- $k_{r_j} \in \mathcal{K}\left(p_{h(j)}\right)$ for all $j \in \{1, \ldots, S+1\}$ where

$$h(j) = 2 - j + 2\lfloor j/2 \rfloor = \begin{cases} 1 & j \text{ is odd} \\ 2 & j \text{ is even,} \end{cases}$$

- $\ell_{s_j} \in \mathcal{L}_{p_{h(j)}, k_{r_j}}$ and $\ell_{s_j} \in \mathcal{L}_{p_{h(j+1)}, k_{r_{j+1}}}$ for all $j \in \{1, \ldots, S\}$,

and either

$$S \text{ is even, } h(1) = h(S+1) \text{ and } r_1 = r_{S+1} \tag{9}$$

or

$$v_{p_{h(1)}, k_{r_1}}, v_{p_{h(S+1)}, k_{r_{S+1}}} \notin \mathbb{Z}, \tag{10a}$$

$$\left\{\ell \in \mathcal{L}_{p_{h(1)}, k_{r_1}} \,:\, z_\ell \in (0, 1)\right\} = \{\ell_{s_1}\}, \tag{10b}$$

$$\left\{\ell \in \mathcal{L}_{p_{h(S+1)}, k_{r_{S+1}}} \,:\, z_\ell \in (0, 1)\right\} = \{\ell_{s_S}\}. \tag{10c}$$

If (9) holds, let $\varepsilon = \min_{i=1}^{S} \left\{z_{\ell_{s_j}}, 1 - z_{\ell_{s_j}}\right\} > 0$, $\underline{\boldsymbol{z}}$ and $\overline{\boldsymbol{z}}$ be such that

$$\underline{z}_{\ell_i} = \begin{cases} z_{\ell_i} - \varepsilon & \text{if } i = s_j \text{ for an odd } j \in S, \\ z_{\ell_i} + \varepsilon & \text{if } i = s_j \text{ for an even } j \in S, \\ z_{\ell_i} & \text{otherwise} \end{cases} \tag{11a}$$

$$\overline{z}_{\ell_i} = \begin{cases} z_{\ell_i} + \varepsilon & \text{if } i = s_j \text{ for an odd } j \in S, \\ z_{\ell_i} - \varepsilon & \text{if } i = s_j \text{ for an even } j \in S, \\ z_{\ell_i} & \text{otherwise.} \end{cases} \tag{11b}$$

The result follows from Lemma 6.1 (with $\underline{\boldsymbol{v}} = \overline{\boldsymbol{v}} = \boldsymbol{v}$) because $\sum_{\ell \in \mathcal{L}_{p,k}} \underline{z}_\ell = \sum_{\ell \in \mathcal{L}_{p,k}} \overline{z}_\ell = \sum_{\ell \in \mathcal{L}_{p,k}} z_\ell$ for all $p \in \mathcal{P}$ and $k \in \mathcal{K}(p)$.

If instead (10) holds let

$$\varepsilon = \min \left\{ \min_{i=1}^{S} \left\{ z_{\ell_{s_j}}, 1 - z_{\ell_{s_j}} \right\}, v_{p_{h(1)},k_{r_1}}, v_{p_{h(S+1)},k_{r_{S+1}}} \right\}$$

$\underline{\boldsymbol{z}}$ and $\overline{\boldsymbol{z}}$ be as defined in (11) with this new $\varepsilon$, and the result again follows from Lemma 6.1 by noting that (10a) implies $\varepsilon$ remains strictly positive.

For case 2 we only need to prove the result for $P \geq 3$. By (8) (and the assumption that at least one $z_l$ is fractional), there exist $k \in \mathcal{K}(p_P)$ such that either $|\{\ell \in \mathcal{L}_{p_P,k} \,:\, z_\ell \in (0,1)\}| \geq 2$ or, $|\{\ell \in \mathcal{L}_{p_P,k} \,:\, z_\ell \in (0,1)\}| = 1$ and $v_{p_P,k} > 0$.

In the first case let $s_1 \neq s_2$ be such that $\ell_{s_1}, \ell_{s_2} \in \{\ell \in \mathcal{L}_{p_P,k} \,:\, z_\ell \in (0,1)\}$, $\varepsilon = \min_{i=1}^{2} \left\{ z_{\ell_{s_j}}, 1 - z_{\ell_{s_j}} \right\} > 0$, and $\underline{\boldsymbol{z}}$ and $\overline{\boldsymbol{z}}$ be such that

$$\underline{z}_{\ell_i} = \begin{cases} z_{\ell_i} - \varepsilon & \text{if } i = s_1, \\ z_{\ell_i} + \varepsilon & \text{if } i = s_2, \\ z_{\ell_i} & \text{otherwise} \end{cases}$$

$$\overline{z}_{\ell_i} = \begin{cases} z_{\ell_i} + \varepsilon & \text{if } i = s_1, \\ z_{\ell_i} - \varepsilon & \text{if } i = s_2, \\ z_{\ell_i} & \text{otherwise.} \end{cases}$$

Then $\sum_{\ell \in \mathcal{L}_{p_P,k}} \underline{z}_\ell = \sum_{\ell \in \mathcal{L}_{p_P,k}} \overline{z}_\ell = \sum_{\ell \in \mathcal{L}_{p_P,k}} z_\ell$ for all $k \in \mathcal{K}(p_P)$. Furthermore, by assumption, for all $k \in \mathcal{K}(p_P)$ and $i \in \{1, \ldots, P-1\}$ there exist $k' \in \mathcal{K}(p_i)$ such that $\mathcal{L}_{p_P,k} \subseteq \mathcal{L}_{p_i,k'}$, so we also have $\sum_{\ell \in \mathcal{L}_{p,k}} \underline{z}_\ell = \sum_{\ell \in \mathcal{L}_{p,k}} \overline{z}_\ell = \sum_{\ell \in \mathcal{L}_{p,k}} z_\ell$ for all $p \in \mathcal{P}$ and $k \in \mathcal{K}(p)$. The result then follows from Lemma 6.1 (with $\underline{\boldsymbol{v}} = \overline{\boldsymbol{v}} = \boldsymbol{v}$).

In the second case, let $s \in \{1, \ldots, L\}$ be such that $\{\ell_s\} = \{\ell \in \mathcal{L}_{p_P,k} \,:\, z_\ell \in (0,1)\}$,

$$\varepsilon = \min \left\{ z_{\ell_s}, 1 - z_{\ell_s}, \min \left\{ \min \left\{ v_{p,k'}, 1 - v_{p,k'} \right\} \,:\, p \in \mathcal{P}, \quad k' \in \mathcal{K}(p) \text{ such that } \ell_s \in \mathcal{L}_{p,k'} \right\} \right\},$$

and $\underline{\boldsymbol{z}}$ and $\overline{\boldsymbol{z}}$ be such that

$$\underline{z}_{\ell_i} = \begin{cases} z_{\ell_i} - \varepsilon & \text{if } i = s, \\ z_{\ell_i} & \text{otherwise} \end{cases}$$
$$\overline{z}_{\ell_i} = \begin{cases} z_{\ell_i} + \varepsilon & \text{if } i = s, \\ z_{\ell_i} & \text{otherwise.} \end{cases}$$

By assumption, for all $i \in \{1, \ldots, P-1\}$ there exist $k' \in \mathcal{K}(p_i)$ such that $\mathcal{L}_{p_P,k} \subseteq \mathcal{L}_{p_i,k'}$ and $\mathcal{T}_{p_P,k} \subseteq \mathcal{T}_{p_i,k'}$ so, $v_{p_P,k} > 0$ implies that $v_{p,k'} > 0$ for all $p \in \mathcal{P}$ and $k' \in \mathcal{K}(p)$ such that $\ell_s \in \mathcal{L}_{p,k'}$. Then $\varepsilon > 0$ and the result follows from Lemma 6.1. □

*Proof of Lemma 4.3.* Let

$$\begin{aligned} \mathbf{x}_1 &= (k_1, k_1, k_1), \quad \mathbf{x}_2 = (k_3, k_3, k_3), \\ \mathbf{x}_1 &= (k_1, k_2, k_3), \quad \mathbf{x}_2 = (k_3, k_2, k_1), \\ \mathbf{x}_1 &= (k_2, k_1, k_2), \quad \mathbf{x}_2 = (k_2, k_3, k_2), \end{aligned}$$

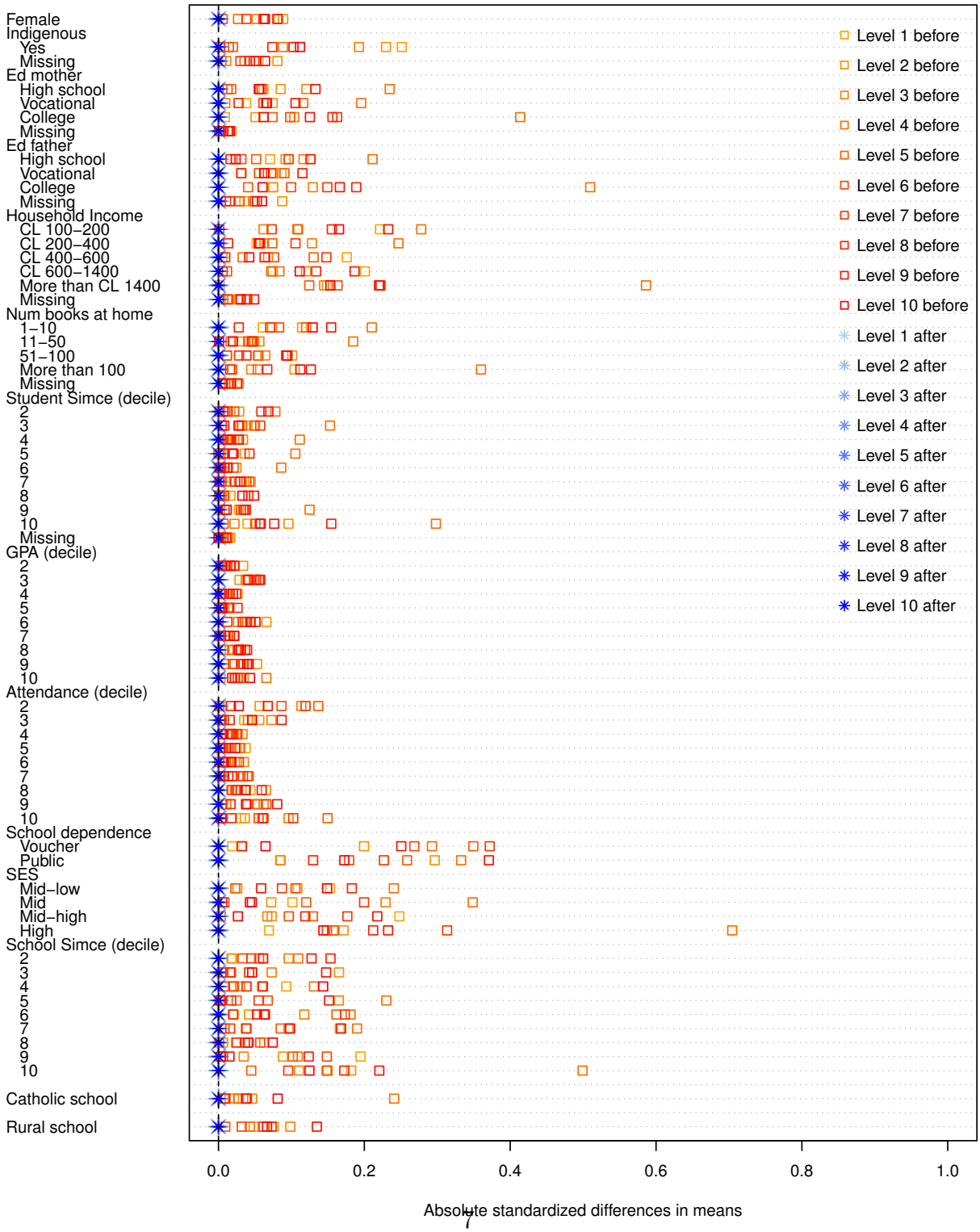
$T = 3$, $L = 6$ and $N_{p,k} = 1$ for all $p$ and $k$. The feasible region of the LP relaxation of (4) for this case is given by

$$\begin{aligned} &|z_{\ell_1} + z_{\ell_3} - 1| \le v_{p_1,k_1}, \quad |z_{\ell_5} + z_{\ell_6} - 1| \le v_{p_1,k_2}, \quad |z_{\ell_2} + z_{\ell_4} - 1| \le v_{p_1,k_3}, \\ &|z_{\ell_1} + z_{\ell_5} - 1| \le v_{p_2,k_1}, \quad |z_{\ell_3} + z_{\ell_4} - 1| \le v_{p_2,k_2}, \quad |z_{\ell_2} + z_{\ell_6} - 1| \le v_{p_2,k_3} \\ &|z_{\ell_1} + z_{\ell_4} - 1| \le v_{p_3,k_1}, \quad |z_{\ell_5} + z_{\ell_6} - 1| \le v_{p_3,k_2}, \quad |z_{\ell_2} + z_{\ell_3} - 1| \le v_{p_3,k_3} \\ &\sum_{i=1}^{6} z_{\ell_i} = 1, \qquad 0 \le z_{\ell_i} \le 1 \qquad \forall i \in \{1, \ldots, 6\}. \end{aligned}$$

Using CDDLib (Fukuda 2001) we can check that this LP has 11 fractional extreme points out of a total of 31. In particular, $z_{\ell_i} = 1/2$ for all $i \in \{1, \ldots, 6\}$ and $v_{p,k} = 0$ for all $p$ and $k$ is one such fractional extreme point. Finally, Proposition 4.1 implies that (1) for this data also fails to be integral. □

# Appendix D: Covariate balance

Figure 3: Standardized differences in means in covariates before and after matching for 10 levels of exposure.



## Appendix E: Effect estimates

Table 7: Effect estimates and 95% confidence intervals for different levels of exposure to the earthquake. The point estimates are contrasts with respect to exposure level 1. The 95% confidence account for multiple comparisons.

(a) 3 exposure levels

| Exposure level | Attendance (%) | PSU score |
|---|---|---|
| 2 | -1.00 | 11.00 |
| | [-2.01,0.00] | [1.90,18.50] |
| 3 | -11.85 | 8.00 |
| | [-12.35,-11.00] | [1.50,14.10] |

(b) 5 exposure levels

| Exposure level | Attendance (%) | PSU score |
|---|---|---|
| 2 | -0.00 | 6.00 |
| | [-1.00,1.00] | [-0.60,14.00] |
| 3 | -2.55 | 6.50 |
| | [-4.00,-1.99] | [-0.10,13.50] |
| 4 | -4.00 | 4.50 |
| | [-5.01,-3.00] | [-3.00,11.00] |
| 5 | -12.95 | 8.50 |
| | [-13.70,-12.09] | [0.90,15.10] |

(c) 10 exposure levels

| Exposure level | Attendance (%) | PSU score |
|---|---|---|
| 2 | 0.00 | 4.00 |
| | [-0.01,1.01] | [-3.50,11.60] |
| 3 | 0.00 | 9.00 |
| | [-1.00,1.01] | [0.90,17.10] |
| 4 | -1.00 | 10.00 |
| | [-2.00,0.01] | [2.00,17.60] |
| 5 | -2.00 | 9.00 |
| | [-2.16,-0.99] | [0.90,16.50] |
| 6 | -3.00 | 7.50 |
| | [-4.01,-2.00] | [-0.50,14.50] |
| 7 | -2.00 | -3.50 |
| | [-3.01,-1.69] | [-3.60,11.00] |
| 8 | -6.10 | -4.00 |
| | [-7.66,-5.00] | [-4.10,11.10] |
| 9 | -11.80 | 9.00 |
| | [-12.85,-10.80] | [1.50,17.50] |
| 10 | -13.75 | 10.50 |
| | [-14.56,-12.94] | [3.0,18.60] |